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STAR–IN–COLORING OF THETA AND PLUS GRAPHS

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ABSTRACT

A digraph $G = (V, E)$ is said to admit star–in–coloring if it satisfies the following two conditions: (i) no path of length three is bicolored (ii) if any path of length two with terminal vertices are of the same color, then the edges must be oriented towards the middle vertex. In this paper we prove that path union of theta graph T_α, open star of theta graph $S(n.T_\alpha)$, one point union for path union of theta graph $P_n^t(t.n.T_\alpha)$, plus graph Pl_n,path union of plus graph Pl_n, open star of plus graph $S(t.Pl_n)$, one point union for path union of plus graph $P_n^t(t.n.Pl_m)$ are star–in–coloring graphs.

KEYWORDS: *Coloring; Star–in–Coloring; Star–in–Chromatic Number; Plus Graph; Theta Graph.*

INTRODUCTION

A proper *coloring* of a graph G is a mapping $f: V \to \{1,2,3,\ldots\}$ such that if $e = v_i v_j \epsilon\, E$, then $f(v_i) \neq f(v_j)$. A *star–coloring* of a graph G is a proper coloring of the graph with the condition that no path of length three (P_4) is bicolored. For standard terminology and notation we follow Harary [3]. The concept of star–coloring of graphs was introduced by Grunbaum [1]. The star–coloring of graphs have been investigated by Fertin et al. [2] and Nesetril et al. [6]. A digraph G is said to be in–coloring if any path of length two with end vertices are of same color, then the edges are always directed towards the middle vertex. Sudha and Kanniga [7, 8] introduced a new concept known as star–in–coloring of graphs. A graph G is said to be *star–in–coloring* graph if the graph G admits both star – coloring and in – coloring.

First we describe the star–in–coloring of a simple graph as shown in Figure 1. Let $v_1, v_2, v_{3,} v_4$ be the vertices, and let the number within the circle indicates that particular color is assigned to that vertex.

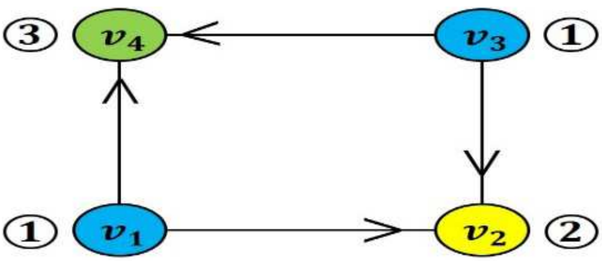

Figure 1: Star – in – Coloring of Cycle C_4

In this graph, we see that no two adjacent vertices have the same color, no path on four vertices is bicolored, further each and every edge in a path of length two in which end vertices have same color are oriented towards the central vertex. Hence it is star–in–colored with orientation. Further the star–in–chromatic number of the above graph is 3.

We [10] proved that cycle, regular cyclic, gear, fan, double fan, web and complete binary tree are star–in–coloring. In [11], proved that splitting of fan graph, as well as double fan graph, Cartesian product of path and a fan, as well as path and a double fan, Tensor product of path and a fan,as well aspath and a double fan are also star–in–coloring graphs. For further results, the interested reader can refer to [4, 5, 9].

- **Definition 1.1** [3] A theta graph is a block with two non-adjacent vertices of degree 3 and all other vertices of degree 2. We denote a theta graph by T_α.
- **Definition 1.2** [5] Take $P_2, P_4, \ldots, P_{n-2}, P_n, P_n, P_{n-2}, \ldots, P_4, P_2$ paths on $2, 4, \ldots, n-2, n, n, n-2, \ldots, 4, 2$ vertices and arrange them centrally horizontal, where $n \equiv 0 \ (mod\ 2), n \neq 2$. A graph obtained by vertical vertices of given successive paths is known as a plus graph of size n and it is denoted by Pl_n.
- **Definition 1.3** [5] Let G be graph and $G_1, G_2, \ldots, G_n, n \geq 2$ be n copies of graph G. Then the graph obtained by adding an edge from $G_i$to $G_{i+1}, (1 \leq i \leq n-1)$ is called path union of G.
- **Definition 1.4** [12] Let G be a graph of n vertices. The graph obtained by replacing each vertex of the $K_{1,n}$ by a copy of G is called a star of G denoted by $S(n. G)$.
- **Definition 1.5** The minimum number of colors required for the star–in–coloring of a graph G is called the *star–in–chromatic number* of G and is denoted by $\chi_{si}(G)$.

MAIN RESULTS

Theorem 1

The path union of theta graph in a path P_n admits star–in–coloring and its star–in–chromatic number is 5.

Proof

Let G be the graph obtained by joining n copies of theta graph by a path P_n, which consists of $8n$ vertices and $10n-1$ edges. We denote v_i^j be the i[th] vertex in the j[th] copy of G, where $1 \leq i \leq 8,\ and\ \ 1 \leq j \leq n$.

We define a function $f: V \to \{1, 2, 3, \ldots\}$ such that $f(v_i) \neq f(v_j)$ if $v_i v_j \epsilon E$, as follows:

For each $j = 1,2, \ldots, n$, we have

$$f(v_i^j) = \begin{cases} 1, & if\ i \equiv 1\ (mod\ 2) \\ \frac{i}{2} + 1, & if\ i \equiv 0\ (mod\ 2) \end{cases}$$

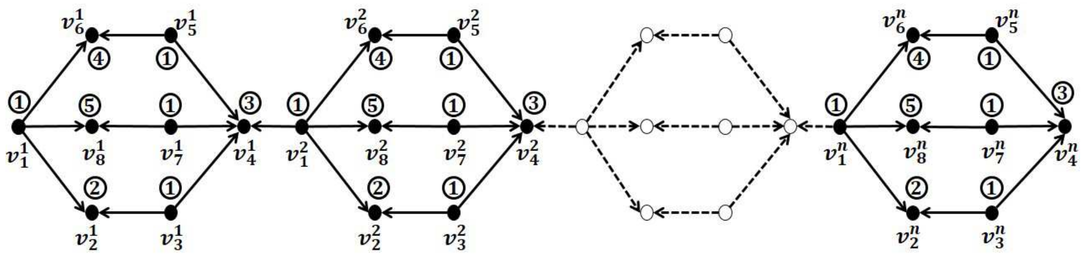

Figure 2: Star-in-Coloring of Path Union of Theta Graph.

In this pattern of coloring, the graph G is star–in–colored and its star–in–chromatic number is 5.

Theorem 2

An open star of n copies of theta graph T_α admits star–in–coloring and its star–in–chromatic number is $\chi_{si}[S(n.T_\alpha)] = 5.$

Proof

Let G be a graph obtained by replacing each vertices of $K_{1,n}$ except the central vertex of $K_{1,n}$ by the theta graph T_α, i.e., $G = S(n.T_\alpha)$. This graph G consists of $8n + 1$ vertices and $10n$ edges.

Let v_0 be the central vertex of G. Let v_i^j be the i[th] vertex in the j[th] copy of T_α, where $1 \le i \le 8,\ 1 \le j \le n.$ Now we shall join each k[th] vertex of all copies of T_α to the apex vertex v_0, where k is any fixed number between 1 and 8.

We define a function $f: V \to \{1, 2, 3, \dots\}$ such that $f(v_i) \neq f(v_j)$ if $v_i v_j \epsilon E$, as follows:

$$f(v_0) = 2$$

$$f(v_i^j) = \begin{cases} 1, & if\ i \equiv 1\ (mod\ 2) \\ \frac{i}{2} + 1, & if\ i \equiv 0\ (mod\ 2), i > 0 \end{cases}$$

In this pattern of coloring, the graph G is star–in–colored and its star–in–chromatic number is $\chi_{si}[S(n.T_\alpha)] = 5.$

The star–in–coloring of $S(n.T_\alpha)$ is shown in Fig. 3.

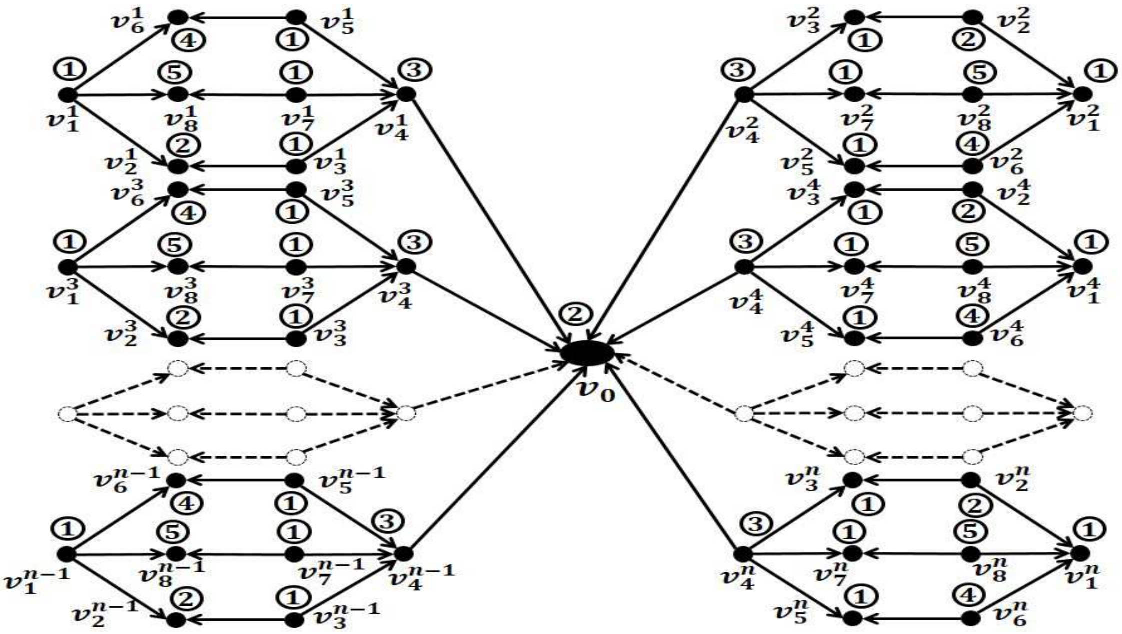

Figure 3: Star-in-Coloring of $S(n.T_\alpha)$.

Theorem 3

One point union for path union of theta graph $P_n^t(t.n.T_\alpha)$ admits star–in–coloring and its star–in–chromatic number is $\chi_{si}[P_n^t(t.n.T_\alpha)] = 5.$

Proof

Let $G = P_n^t(t.n.T_\alpha)$ be a graph obtained by replacing each vertex of P_n^i $(1 \le i \le t)$ by the theta graph T_α, each vertex $v_{i,1}^1(1 \le i \le t)$is connected to the apex vertex v_0, where t is any positive integer. This graph G consists of $8tn + 1$ vertices and $(9n + 1)t + (n - 1)t$ edges.

Let $v_{i,j}^k$ be the k[th] copy of the j[th] vertex in the i[th] row, where $1 \le i \le t,\ 1 \le j \le 8, 1 \le k \le n$.

We define a function $f: V \to \{1, 2, 3, \ldots\}$ such that $f(v_i) \ne f(v_j)$ if $v_i v_j \epsilon\ E$, as follows:

$$f(v_0) = 3$$

$$f(v_{i,j}^k) = \begin{cases} 1, & if\ j \equiv 1\ (mod\ 2) \\ \frac{j}{2} + 1, & if\ j \equiv 0\ (mod\ 2), j > 0 \end{cases}$$

In this pattern of coloring, the graph G is star–in–colored and its star–in–chromatic number is $\chi_{si}[P_n^t(t.n.T_\alpha)] = 5$.

The star–in–coloring of $P_n^t(t.n.T_\alpha)$ is shown in Fig. 4.

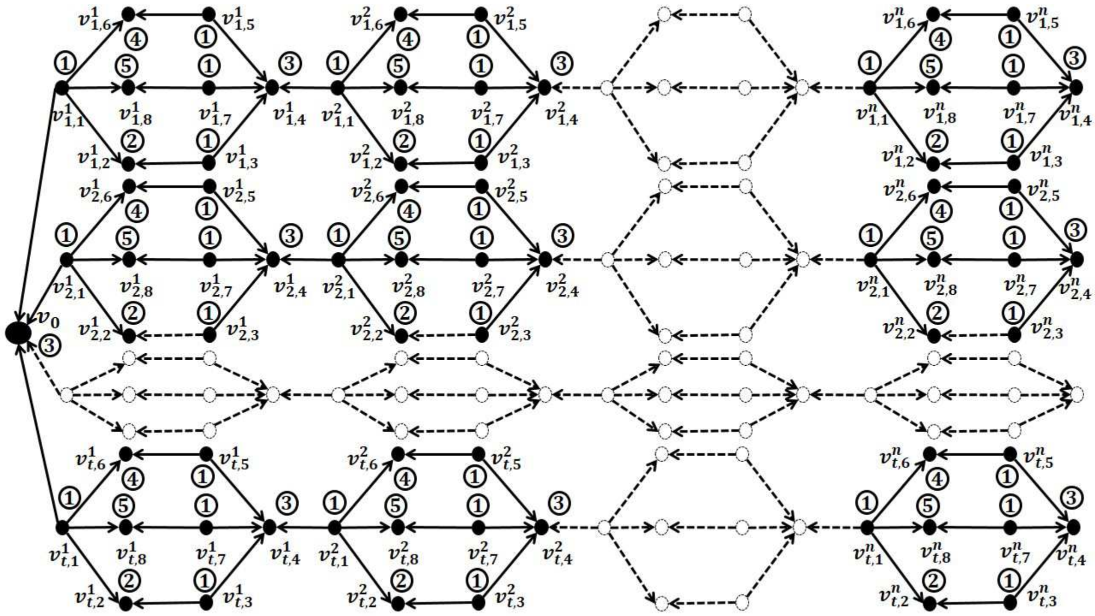

Figure 4: Star-in-Coloring of $P_n^t(t.n.T_\alpha)$

Theorem 4

A plus graph Pl_n admits star–in–coloring and its star–in–chromatic number is$\chi_{si}[Pl_n] = 5$, where $n \equiv 0\ (mod\ 2), n > 2$.

Proof

Let $G = Pl_n$be a graph, which consists of $\frac{n^2}{2} + n$ vertices and n^2edges. We denote $v_{i,j}$ is the j[th] vertex in the i[th] row, where $1 \le i \le \frac{n}{2},\ 1 \le j \le 2i$ and $\frac{n}{2} + 1 \le i \le n, 1 \le j \le 2n - 2i + 2$.

We define a function $f: V \to \{1, 2, 3, \ldots\}$ such that $f(v_i) \neq f(v_j)$ if $v_i v_j \epsilon E$, as follows:

If $i \equiv 1 \ (mod\ 2)$ and $i \leq \frac{n}{2}$

$$f(v_{i,j}) = \begin{cases} 1, & if\ j \equiv 1\ (mod\ 2) \\ 2, & if\ j \equiv 2\ (mod\ 4) \\ 5, & if\ j \equiv 0\ (mod\ 4) \end{cases}$$

If $i \equiv 0\ (mod\ 2)$ and $i \leq \frac{n}{2}$

$$f(v_{i,j}) = \begin{cases} 1, & if\ j \equiv 1\ (mod\ 2) \\ 3, & if\ j \equiv 2\ (mod\ 4) \\ 4, & if\ j \equiv 0\ (mod\ 4) \end{cases}$$

When $i \geq \frac{n}{2} + 1$ the pattern of coloring has been grouped into two cases:

Case 1: When $n \equiv 0\ (mod\ 4)$

Subcase 1.1: For $i \equiv 1\ (mod\ 2)$

$$f(v_{i,j}) = \begin{cases} 1, & if\ j \equiv 0\ (mod\ 2) \\ 2, & if\ j \equiv 1\ (mod\ 4) \\ 5, & if\ j \equiv 3\ (mod\ 4) \end{cases}$$

Subcase 1.2: For $i \equiv 0\ (mod\ 2)$

$$f(v_{i,j}) = \begin{cases} 1, & if\ j \equiv 0\ (mod\ 2) \\ 4, & if\ j \equiv 1\ (mod\ 4) \\ 3, & if\ j \equiv 3\ (mod\ 4) \end{cases}$$

Case 2: When $n \equiv 2\ (mod\ 4)$

Subcase 2.1: For $i \equiv 1\ (mod\ 2)$

$$f(v_{i,j}) = \begin{cases} 1, & if\ j \equiv 0\ (mod\ 2) \\ 5, & if\ j \equiv 1\ (mod\ 4) \\ 2, & if\ j \equiv 3\ (mod\ 4) \end{cases}$$

Subcase 2.2: For $i \equiv 0\ (mod\ 2)$

$$f(v_{i,j}) = \begin{cases} 1, & if\ j \equiv 0\ (mod\ 2) \\ 3, & if\ j \equiv 1\ (mod\ 4) \\ 4, & if\ j \equiv 3\ (mod\ 4) \end{cases}$$

From the above cases, we conclude that the plus graph Pl_n is star–in–colored and its star–in–chromatic number is 5.

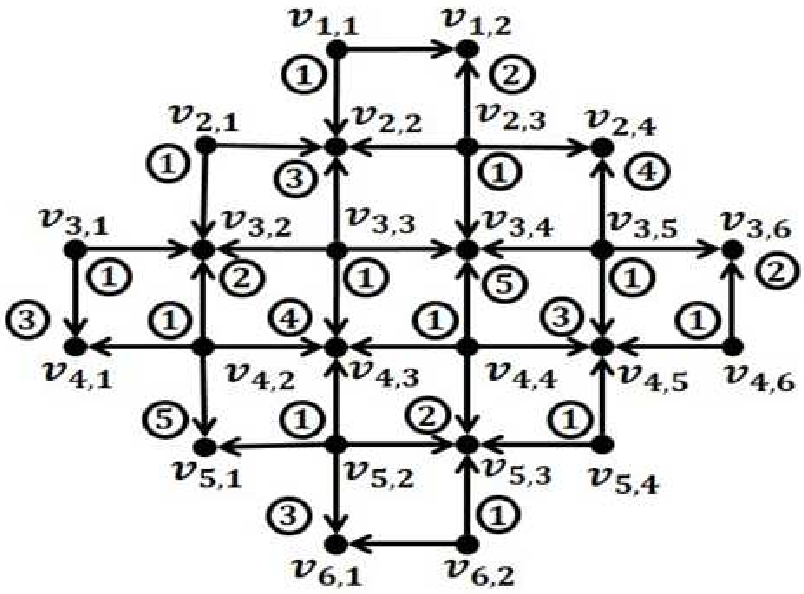

Figure 5: Star-in-Coloring of Pl_6

The star–in–coloring of Pl_6 is shown in Fig. 5.

Theorem 5

Path union of the plus graph Pl_n admits star–in–coloring and its star–in–chromatic number is 5,where $n \equiv 0\ (mod\ 4)$.

Proof

Let G be a path union of r copies of the plus graph Pl_n, which consists of $\left(\frac{n^2}{2}+n\right)r$ vertices and rn^2+r-1 edges. We denote $v_{i,j}^k$ is the j[th] vertex in the i[th] row of k[th] copy of Pl_n, where $1 \le i \le \frac{n}{2}$, $1 \le j \le 2i$ and $\frac{n}{2}+1 \le i \le n, 1 \le j \le 2n-2i+2$ and $1 \le k \le r$. Join the vertices $v_{\frac{n}{2}+1,n}^k$ to $v_{\frac{n}{2}+1,1}^{k+1}$ for $k = 1,2,\dots,r-1$ by an edge.

We define a function $f: V \to \{1,2,3,\dots\}$ such that $f(v_i) \neq f(v_j)$ if $v_iv_j \epsilon\ E$, as follows:

If $i \equiv 1\ (mod\ 2)$ and $i \le \frac{n}{2}, \forall k$

$$f\left(v_{i,j}^k\right) = \begin{cases} 1, & if\ j \equiv 1\ (mod\ 2) \\ 2, & if\ j \equiv 2\ (mod\ 4) \\ 5, & if\ j \equiv 0\ (mod\ 4) \end{cases}$$

If $i \equiv 0\ (mod\ 2)$ and $i \le \frac{n}{2}, \forall k$

$$f\left(v_{i,j}^k\right) = \begin{cases} 1, & if\ j \equiv 1\ (mod\ 2) \\ 3, & if\ j \equiv 2\ (mod\ 4) \\ 4, & if\ j \equiv 0\ (mod\ 4) \end{cases}$$

If $i \equiv 1\ (mod\ 2)$ and $i \ge \frac{n}{2}+1, \forall k$

$$f(v_{i,j}^k) = \begin{cases} 1, & if\ j \equiv 0\ (mod\ 2) \\ 2, & if\ j \equiv 1\ (mod\ 4) \\ 5, & if\ j \equiv 3\ (mod\ 4) \end{cases}$$

If $i \equiv 0\ (mod\ 2)$ and $i \geq \frac{n}{2} + 1, \forall k$

$$f(v_{i,j}^k) = \begin{cases} 1, & if\ j \equiv 0\ (mod\ 2) \\ 4, & if\ j \equiv 1\ (mod\ 4) \\ 3, & if\ j \equiv 3\ (mod\ 4) \end{cases}$$

In this pattern of coloring, the plus graph G is star–in–colored and its star–in–chromatic number is 5.

The star–in–coloring of path union of r copies of Pl_4 is shown in Fig. 6.

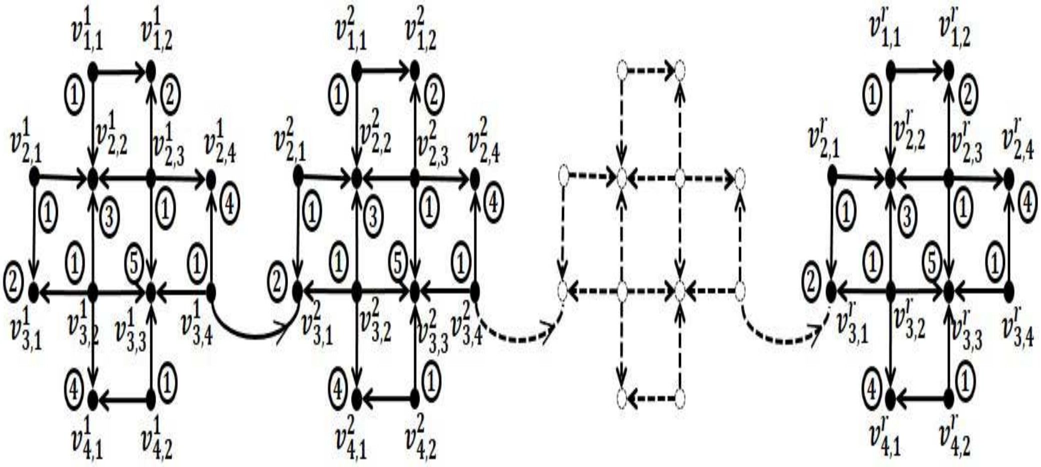

Figure 6: Star-in-Coloring of Path Union of r Copies of Pl_4

Theorem 6

An open star of t copies of plus graph $S(t.Pl_n)$ admits star–in–coloring and its star–in–chromatic number is $\chi_{si}[S(t.Pl_n)] = 5$, where $n \equiv 0\ (mod\ 2), n > 2$.

Proof

Let $G = S(t.Pl_n)$ be a graph obtained by replacing each vertices of $K_{1,t}$ except the apex vertex of $K_{1,t}$by the graph Pl_n. This graph G consists of $t\left(\frac{n^2}{2} + n\right) + 1$ vertices and $t(n^2 + 1)$ edges.

Let v_0 be the apex vertex of $K_{1,t}$. We denote $v_{i,j}^k$ is the j[th] vertex in the i[th] row of k[th] copy of Pl_n in$G$, where $1 \leq i \leq \frac{n}{2}$, $1 \leq j \leq 2i$ and $\frac{n}{2} + 1 \leq i \leq n, 1 \leq j \leq 2n - 2i + 2$ and $1 \leq k \leq t$.Join the vertices $v_{\frac{n}{2}+1,1}^k$ with the vertex v_0 by an edge,$k = 1,2, \dots, t$.

We define a function $f: V \to \{1,2,3, \dots\}$ such that $f(v_i) \neq f(v_j)$ if $v_i v_j \epsilon\ E$, as follows:

If $i \equiv 1\ (mod\ 2)$ and $i \leq \frac{n}{2}, \forall k$

$$f(v_{i,j}^k) = \begin{cases} 1, & if\ j\ \equiv 1\ (mod\ 2) \\ 2, & if\ j\ \equiv 2\ (mod\ 4) \\ 5, & if\ j\ \equiv 0\ (mod\ 4) \end{cases}$$

If $i \equiv 0\ (mod\ 2)$ and $i \leq \frac{n}{2}, \forall k$

$$f(v_{i,j}^k) = \begin{cases} 1, & if\ j\ \equiv 1\ (mod\ 2) \\ 3, & if\ j\ \equiv 2\ (mod\ 4) \\ 4, & if\ j\ \equiv 0\ (mod\ 4) \end{cases}$$

When $i \geq \frac{n}{2} + 1$ the pattern of coloring has been grouped into two cases:

Case 1: When $n \equiv 0\ (mod\ 4)$

Subcase 1.1: For $i \equiv 1\ (mod\ 2), \forall k$

$$f(v_{i,j}^k) = \begin{cases} 1, & if\ j\ \equiv 0\ (mod\ 2) \\ 2, & if\ j\ \equiv 1\ (mod\ 4) \\ 5, & if\ j\ \equiv 3\ (mod\ 4) \end{cases}$$

Subcase 1.2: For $i \equiv 0\ (mod\ 2), \forall k$

$$f(v_{i,j}^k) = \begin{cases} 1, & if\ j\ \equiv 0\ (mod\ 2) \\ 4, & if\ j\ \equiv 1\ (mod\ 4) \\ 3, & if\ j\ \equiv 3\ (mod\ 4) \end{cases}$$

Case 2: When $n \equiv 2\ (mod\ 4)$

Subcase 2.1: For $i \equiv 1\ (mod\ 2), \forall k$

$$f(v_{i,j}^k) = \begin{cases} 1, & if\ j\ \equiv 0\ (mod\ 2) \\ 5, & if\ j\ \equiv 1\ (mod\ 4) \\ 2, & if\ j\ \equiv 3\ (mod\ 4) \end{cases}$$

Subcase 2.2: For $i \equiv 0\ (mod\ 2), \forall k$

$$f(v_{i,j}^k) = \begin{cases} 1, & if\ j\ \equiv 0\ (mod\ 2) \\ 3, & if\ j\ \equiv 1\ (mod\ 4) \\ 4, & if\ j\ \equiv 3\ (mod\ 4) \end{cases}$$

and $f(v_0) = 3$

From the above cases, we conclude that thegraph G is star–in–colored and its star–in–chromatic number is 5.

The star–in–coloring of $S(t.Pl_4)$ is shown in Fig. 7.

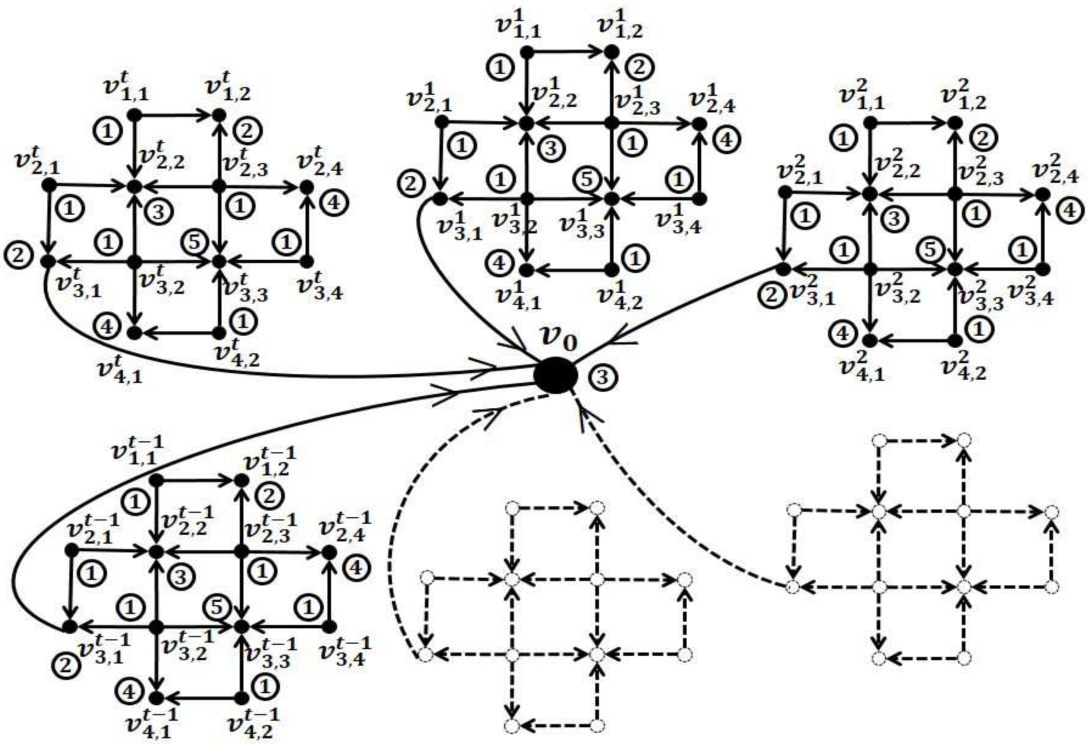

Figure 7: Star-in-Coloring of $S(t.Pl_4)$

Theorem 7

One point union for path union of plus graph $P_n^t(t.n.Pl_m)$ admits star–in–coloring and its star–in–chromatic number is $\chi_{si}[P_n^t(t.n.Pl_m)] = 5$, where $m \equiv 0\ (mod\ 4)$.

Proof

Let $G = P_n^t(t.n.Pl_m)$ be a graph obtained by replacing each vertices of $K_{1,t}$ except the apex vertex of $K_{1,t}$ by the path union of n copies of the graph Pl_m. This graph G consists of $tn\left(\frac{m^2}{2}+m\right)+1$ vertices and $tn(m^2+1)$ edges.

Let v_0 be the apex vertex of $K_{1,t}$ with t branches. We denote $v_{i,j}^{s,k}$ is the j^th vertex in the i^th row of k^th copy of Pl_m lies in the s^th branch of the graph G, where $1 \le i \le \frac{m}{2}$, $1 \le j \le 2i$ and $\frac{m}{2}+1 \le i \le m, 1 \le j \le 2m-2i+2$ and $1 \le k \le n, 1 \le s \le t$. Join the vertices $v_{\frac{m}{2}+1,m}^{s,k}$ to $v_{\frac{m}{2}+1,1}^{s,k+1}$ for $k = 1,2,\dots,n-1, s = 1,2,\dots,t$ by an edge. Also join the vertices of $v_{\frac{m}{2}+1,1}^{s,1}$ with v_0 by an edge.

We define a function $f: V \to \{1,2,3,\dots\}$ such that $f(v_i) \neq f(v_j)$ if $v_i v_j \epsilon E$, as follows:

If $i \equiv 1\ (mod\ 2)$ and $i \le \frac{n}{2}, \forall\ s,k$

$$f(v_{i,j}^{s,k}) = \begin{cases} 1, & if\ j \equiv 1\ (mod\ 2) \\ 2, & if\ j \equiv 2\ (mod\ 4) \\ 5, & if\ j \equiv 0\ (mod\ 4) \end{cases}$$

If $i \equiv 0\ (mod\ 2)$ and $i \le \frac{n}{2}, \forall\ s,k$

$$f\left(v_{i,j}^{s,k}\right) = \begin{cases} 1, & if\ j \equiv 1\ (mod\ 2) \\ 3, & if\ j \equiv 2\ (mod\ 4) \\ 4, & if\ j \equiv 0\ (mod\ 4) \end{cases}$$

If $i \equiv 1\ (mod\ 2)$ and $i \geq \frac{n}{2} + 1, \forall\ s, k$

$$f\left(v_{i,j}^{s,k}\right) = \begin{cases} 1, & if\ j \equiv 0\ (mod\ 2) \\ 2, & if\ j \equiv 1\ (mod\ 4) \\ 5, & if\ j \equiv 3\ (mod\ 4) \end{cases}$$

If $i \equiv 0\ (mod\ 2)$ and $i \geq \frac{n}{2} + 1, \forall\ s, k$

$$f\left(v_{i,j}^{s,k}\right) = \begin{cases} 1, & if\ j \equiv 0\ (mod\ 2) \\ 4, & if\ j \equiv 1\ (mod\ 4) \\ 3, & if\ j \equiv 3\ (mod\ 4) \end{cases}$$

and $f(v_0) = 3$

In this pattern of coloring, the graph G is star–in–colored and its star–in–chromatic number is 5.

The star–in–coloring of $P_n^t(t.n.Pl_4)$ is shown in Fig. 8.

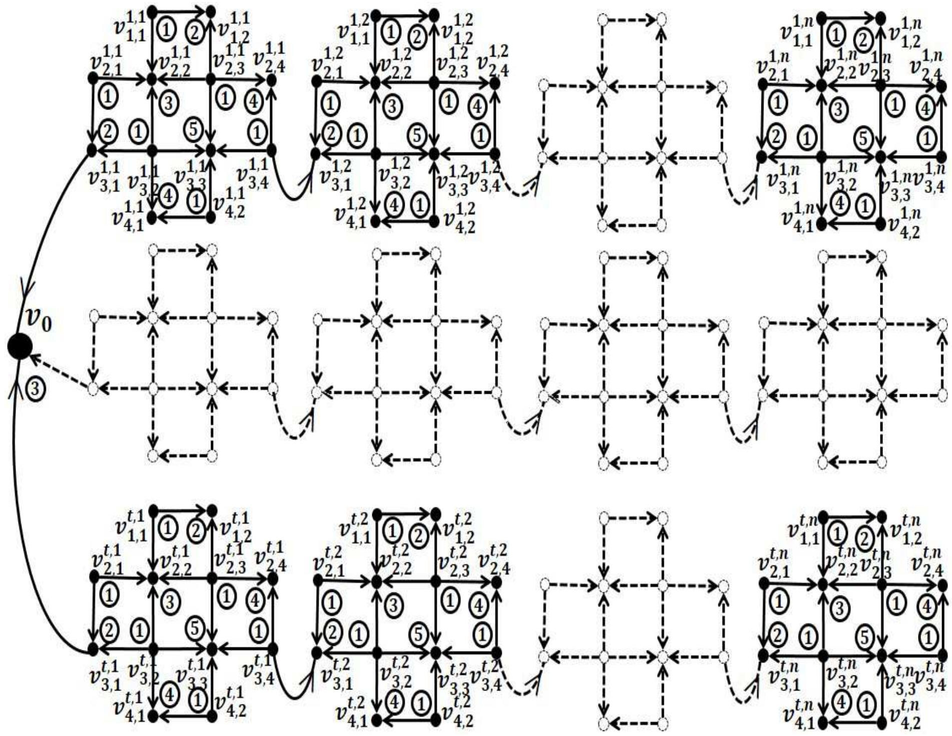

Figure 8: Star-in-Coloring of $P_n^t(t.n.Pl_4)$

CONCLUSION

In this paper we investigate the star–in–coloring concept to the graphs related to theta and plus graphs. Further we find the star–in–chromatic number of these graphs. It is interesting to note that each and every graph discussed in this paper has a star–in–chromatic number five. To find a new class of graphs with star–in–chromatic number five, is an interesting open area of research.

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